Last Time: Uniqueness of RREF

Thm: RREF's are uniquely determined. ← WTS

we've shown (up to now):

① Elementary row ops are "reversible"

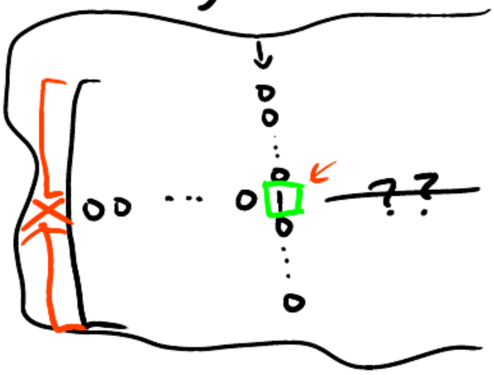
↳ "row equivalence" is an equivalence relation.

② Linear Combination Lemma

↳ If A row-reduces to B, then rows of A are lin. comb. of rows of B.

Lem: If M is in RREF, then nonzero rows of M are not linear combinations of the other rows.

pf: Let M be a matrix in RREF. Every nonzero row of M has a leading 1. Furthermore, all leading 1's are the only nonzero entries in their column.



In particular, every linear combination of the other rows has 0 in the column corresponding to any given leading 1; hence that row is not a lin. comb. of the other rows (they don't match in that coord!) ☒

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pf (Uniqueness of RREF): Let M be a matrix with m rows. We proceed by induction on the number of columns of M.

<u>Base Case</u>: If M has only 1 column, either all entries of this column are 0 or not. If all entries of the column are 0, then M is in RREF. Otherwise, this column has a nonzero entry. Swap any such entry to the first position, multiply by a suitable nonzero scalar, and finally eliminate all other entries. The result is an $m \times 1$ matrix with 1 in the first entry and 0's in all other entries. Hence M has a unique RREF in these cases.

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{all rows are zero!}$$

$$k \neq 0 \quad \begin{bmatrix} \vdots \\ k \\ \vdots \end{bmatrix} \rightsquigarrow \begin{bmatrix} k \\ \vdots \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 \\ \vdots \end{bmatrix} \rightsquigarrow \begin{array}{r} \rho_1 \rightarrow \\ \\ \rho_i \rightarrow \end{array} \begin{bmatrix} 1 \\ \vdots \\ a \\ \vdots \end{bmatrix} \xrightarrow{\rho_i - a\rho_1} \begin{bmatrix} 1 \\ \vdots \\ 0 \\ \vdots \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

<u>Induction Step</u>: Suppose M has $n+1$ columns and suppose every $m \times n$ matrix has a unique RREF. Suppose M has two RREFs, B and C. Because A is an $m \times n$ matrix, our assumption yields B and C have the same first n columns (because our RREFs for M contain an RREF for A). Consider the homogeneous linear systems determined by B and C (i.e. $B\vec{x} = \vec{0}$ and $C\vec{x} = \vec{0}$). If $B \neq C$, they differ in the last column, so

$$M = \big[\underbrace{A}_{n \text{ columns}} \,\big|\, \vec{a}\,\big]$$

$$B = \left[\operatorname{rref}(A) \,\middle|\, \vec{b}\,\right]$$

$$C = \left[\operatorname{rref}(A) \,\middle|\, \vec{c}\,\right]$$

we could find a row i so that $\underbrace{b_i \neq c_i}$

(where $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$ and $\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$). Either row i has a leading 1 in rref (A) or it is an all-zeros row for rref(A). We may subtract row i of B from row i of C. In the corresponding linear systems, we obtain the equation $(c_i - b_i)x_n = 0$. Thus either $\underbrace{c_i - b_i = 0}$ or $x_n = 0$. As $b_i \neq c_i$, we must have $\boxed{x_n = 0}$ in the solution of this linear system, thus row i must have a leading 1 in column n (b/c x_n is not a free variable). Hence there is exactly one entry in column n which is nonzero. This leading 1 must occur in exactly the same position in both B and C because of the RREF ordering on rows w/ leading 1's. Hence $B = C$ is the unique RREF for M (which is what we wanted ☺). ▨

Point: Every matrix is row-equivalent to a unique matrix in RREF.

Cor: A matrix A and matrix B are row-equivalent if and only if $\text{rref}(A) = \text{rref}(B)$.

Ex: Which of these matrices are row-equivalent?

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 6 \\ 4 & 10 \end{bmatrix} \quad E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}$$

Sol: Compute RREF for each:

$$A: \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \xrightarrow{\rho_2 - 2\rho_1} \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \xrightarrow{-\frac{1}{2}\rho_2} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow{\rho_1 - 3\rho_2} \underbrace{\overset{\text{rref}(A)}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}}$$

$$B: \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix} \xrightarrow{\rho_2 - 2\rho_1} \begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix} = \text{rref}(B)$$

$$C: \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}\rho_2} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \xrightarrow{\rho_2 - \rho_1} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \xrightarrow{\rho_1 + \rho_2} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \text{rref}(C)$$

$$D: \begin{bmatrix} 2 & 6 \\ 4 & 10 \end{bmatrix} \xrightarrow{\rho_2 - 2\rho_1} \begin{bmatrix} 2 & 6 \\ 0 & -2 \end{bmatrix} \xrightarrow[-\frac{1}{2}\rho_2]{\frac{1}{2}\rho_1} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow{\rho_1 - 3\rho_2} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \text{rref}(D)$$

$$E: \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \xrightarrow{-\rho_2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_2} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \text{rref}(E)$$

$$F: \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix} \xrightarrow[\frac{1}{2}\rho_2]{\frac{1}{3}\rho_1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{\rho_2 - \rho_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \text{rref}(F)$$

We have $\text{rref}(A) = \text{rref}(C) = \text{rref}(D) = \text{rref}(E) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So A, C, D, E are row equivalent.

OTOH, $\text{rref}(B) = \begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix}$ and $\text{rref}(F) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, so these are inequivalent to the others on our list. ☒

weakly reduced: $m \left[\overset{n}{A} \right] \vec{x} = \vec{0}$

If $m < n$, then this system has infinitely many solutions.

Ex: Write down all possible 2×3 linear systems (homogeneous) up to row equivalence.

Sol: We give all RREF 2×3 matrices below. for $a,b,c \in \mathbb{R}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & a \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}, \begin{bmatrix} 1 & a & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, every homogeneous 2×3 linear system has the same solution set as $A\vec{x} = \vec{0}$ for one of the matrices A listed above. $\boxtimes$

Linear Maps (determined by matrices)

Defn: A function $L: \mathbb{R}^n \to \mathbb{R}^m$ is linear when $L(\vec{u} + a\vec{v}) = L(\vec{u}) + aL(\vec{v})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $a \in \mathbb{R}$.

Ex: $L: \mathbb{R}^2 \to \mathbb{R}$ defined by $L\begin{bmatrix} x \\ y \end{bmatrix} = x + y$ is a linear map. Indeed, given $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$ and $c \in \mathbb{R}$, we have:

$$L\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + a\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = L\begin{bmatrix} x_1 + ax_2 \\ y_1 + ay_2 \end{bmatrix} = (x_1 + ax_2) + (y_1 + ay_2)$$

$$= (x_1 + y_1) + a(x_2 + y_2)$$

$$= L\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + aL\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}.$$ ▨

Non-ex: $L : \mathbb{R}^1 \to \mathbb{R}^1$ defined by $L[x] = [x^2]$ is not a linear map. To show this, we must find $[x], [y] \in \mathbb{R}^1$ and $a \in \mathbb{R}$ s.t. $L([x] + a[y]) \neq L[x] + aL[y]$.

Trying $a = x = y = 1$, we see

$$L([1] + 1[1]) = L[2] = [4]$$ whereas

$$\neq$$

$$L[1] + 1L[1] = [1] + [1] = [2]$$

So we've verified L is not linear... ▨